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A q-enumeration of directed diagonally convex polyominoes

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Abstract

We enumerate directed diagonally convex polyominoes according to area. Our approach partly goes column by column, and partly goes row by row. In the end, we obtain fairly nice formulas.

Keywords: Directed diagonally convex polyominoes; Enumeration; Area; Simpler formula; Escaliers; Floorsitters; Decomposition

1. Introduction

Directed diagonally convex (ddc)-polyominoes originate from (and are, in fact, equivalent to) so-called *fully directed compact* (*fdc-*) *lattice animals*. These fdc-animals are young: physicists Bhat et al. introduced them in 1986 [1].

The early publications Bhat et al. [1] and Privman and Švrakić [15] are focused on the number (say p_n) of fdc-animals with cardinality n. In [1], the authors argue that p_n is asymptotically equal to λ^n, where $\lambda = 2.66185 \pm 0.00005$. In [15], the function $D = \sum_{n\geqslant 1} p_n q^n$ is derived exactly for the first time. (This D is, in fact, the area generating function for ddc-polyominoes.)

Counting ddc-polyominoes by perimeter was first undertaken by Delest and Fédou [5]. Let r_k be the number of ddc-polyominoes with *site perimeter* $k+1$ (that is to say, with k diagonals). One of the results of [5] is that r_k equals the number of ternary trees with k internal nodes. That is,

$$r_k = \frac{1}{3k+1}\binom{3k+1}{k}. \tag{1}$$

By now, this nice fact about r_k has been established in several different ways: in [5], there are the original algebraic-language proof as well as a bijective proof. Then there

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are two other bijective proofs: an earlier one by Penaud [13], and a more recent one by Svrtan and Feretić [9]. (In [9], the bijection is relatively simple, and is no longer defined recursively.) Then there is—also in [9]—a proof based on Raney's generalized lemma [11, p. 348].

Further, in one of the probabilistic *percolation models*, the first correction term is precisely the number of ddc-polyominoes with k diagonals. Having realized this fact, Bousquet-Mélou [3] and Inui et al. [12] gave yet two different derivations of (1).

Ddc-polyominoes have an interesting "companion". It is the generating function (gf) D, whose variables are the following: d is the diagonals, x the horizontal semiperimeter, and y the vertical semiperimeter. The function D is algebraic, and satisfies the equation

$$D = d(D+1)(D+x)(D+y). \tag{2}$$

Eq. (2) first appeared in Svrtan and Feretić [9]. Then, on pp. 61 and 62 of her habilitation thesis [4], Bousquet-Mélou derived (2) in a new way. Namely, she took the approach called *object grammars* [6].

Let us now return to *q-enumeration* (that is, to enumeration by both area and some—or none—other parameters). It was shown in [9], and was integrated with some corollaries in Feretić [7], that the q-enumeration of ddc-polyominoes may be done by applying Gessel's q-Lagrange inversion formula [10]. The resulting formula for the gf then involves both positive and negative powers of q.

Moreover, in [4, pp. 66 and 67], Bousquet-Mélou q-enumerated ddc-polyominoes by a certain method from her previous paper [2].

In the present paper we, too, shall q-enumerate ddc-polyominoes by the method of [2]. This does not mean, however, that we are going to make a copy of [4, pp. 66 and 67]. Indeed, our way of applying the method will be different, and our final result (i.e., expression for the gf) will look simpler than those of [15] and [4, pp. 66 and 67].

Incidentally, our planned enumerations might as well be performed by Svrtan's method [8] for solving the Temperley recurrences [16]. In fact, although the methods of [2] and [8] were developed independently, they are pretty close to each other.

This paper now continues as follows. In Section 2, we state the necessary definitions and conventions. In Sections 3 and 4, we q-enumerate so-called escalier polyominoes, as well as certain close relatives of theirs; the name of those relatives is floorsitters. We are then able to state and solve our new functional equation for ddc-polyominoes, and we do so in Section 5.

2. Definitions and conventions

If c is a closed unit square in the Cartesian plane, and if the vertices of c have integer coordinates, then c is called a *cell*.

Imagine one or more cells which all lie in the same vertical strip of width one. If connected, the union of those cells is called a *column*.

A *row* is a column rotated by 90°.

Let $K_1, \dots, K_r$ $(r \in N)$ be columns. Suppose that, for $i = 2, \dots, r$, it is the case that:

- the bottom cell of K_i is the right neighbor of the bottom cell of K_{i-1},
- compared with K_{i-1}, the column K_i is either lower by one unit, or equally high, or higher by $\geqslant 1$ units.

The union $\bigcup_{i=1}^{r} K_i$ is then an *escalier polyomino* (see Fig. 1).

Incidentally, the term "polyomino escalier"—coined by the Bordeaux group (introduction of [14])—is a bit problematical, because the English word for *escalier* is *staircase*, and the name *staircase polyomino* is commonly used for another object.

Let $c_1, \dots, c_j$ $(j \in N)$ be cells such that c_i $(i = 2, \dots, j)$ is the lower neighbor of the right neighbor of c_{i-1}. Further, let s_0 be the upper neighbor of c_1, and let s_i $(i = 1, \dots, j)$ be the right neighbor of c_i. Then, the union $\bigcup_{i=1}^{j} c_i$ is a *diagonal*, and the union $\bigcup_{i=0}^{j} s_i$ is the *shadow* of that diagonal.

Let $D_1, \dots, D_k$ $(k \in N)$ be diagonals such that D_1 has just one cell, and such that D_i $(i = 2, \dots, k)$ lies in the shadow of D_{i-1}. Then

- the union $P = \bigcup_{i=1}^{k} D_i$ is a *directed diagonally convex polyomino* (a *ddc-polyomino*),
- the only cell of D_1 is the *source cell* of P,
- the cells of D_k are *target cells* of P,
- the diagonals $D_1, D_2, \dots, D_k$ are the *first, second,…* , kth diagonals of P

(see Fig. 1 again).

By the *floor* of a ddc-polyomino P, we mean the horizontal line containing the lower side of P's source cell.

Let P be a ddc-polyomino. If every diagonal of P touches the floor of P, then P is a *floorsitter*.[1]

Let $m \in N$. An *m-floorsitter* is a floorsitter with exactly m target cells.

So far, we have discussed what is an escalier polyomino and a ddc-polyomino, but we have not discussed what *polyomino* is. So let us define it: a *polyomino* is a union of cells which is finite and possesses connected interior. It is easy to verify that our escalier polyominoes and ddc-polyominoes are indeed polyominoes.

Finally, let us state that in this paper we count polyominoes up to translations, as usual.

3. Escalier polyominoes

In what follows, the gf for escaliers, denoted by $E(s)$, will have five variables: x is the horizontal semiperimeter,[2] y is the vertical semiperimeter, q the area, s the height of the first column, and u the height of the last column.

[1] As a matter of fact, floorsitters are nothing but escaliers with one-cell last columns.

[2] This is the same thing as to say "$x =$ the number of columns".

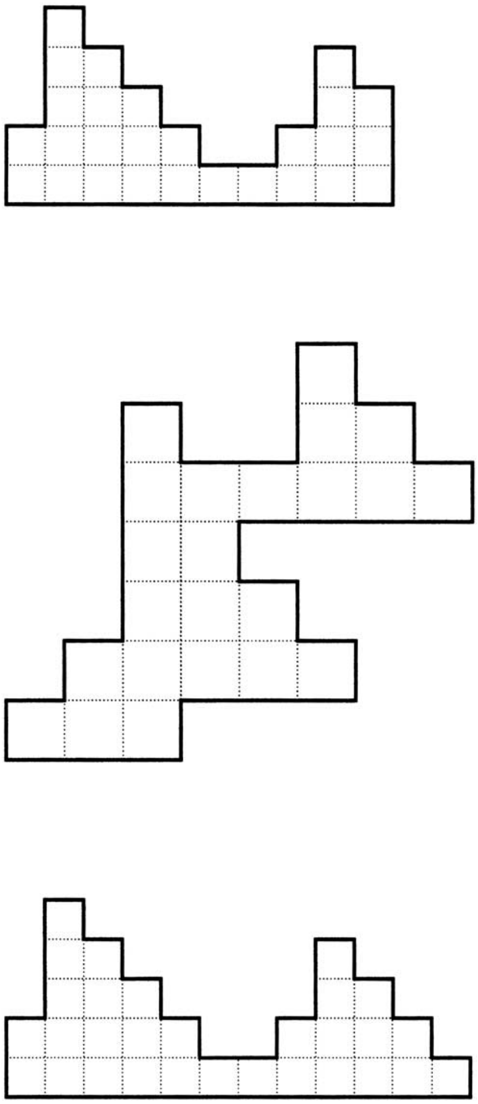

Fig. 1. From top: An escalier polyomino, a directed diagonally convex (ddc-) polyomino, and a floorsitter.

For those of us who have read [2], the following two propositions will be a simple matter. For the rest of us, some related explanations are given in Section 5 of this paper.

Proposition 1. *The gf $E(s)$ satisfies the equation*

$$E(s)=\frac{xyqsu}{1-yqsu}+\frac{xqs}{1-qs}E(1)-\frac{xqs(1-y+yqs)}{1-qs}E(qs). \tag{3}$$

Proposition 2. *The gf for escaliers is given by*

$$E(1) = \frac{\sum_{i=1}^{\infty}(-1)^{i-1}x^i y q^{\binom{i+1}{2}} u \prod_{\ell=1}^{i-1}(1-y+yq^{\ell})/(q)_{i-1}(1-yq^i u)}{\sum_{i=0}^{\infty}(-1)^i x^i q^{\binom{i+1}{2}} \prod_{\ell=1}^{i-1}(1-y+yq^{\ell})/(q)_i}, \tag{4}$$

where the empty product is defined to be one, and where $(q)_0 = 1, (q)_1 = 1-q, (q)_2 = (1-q)(1-q^2)$ *etc.*

4. Floorsitters

As we mentioned in Section 2, a floorsitter is just an escalier with one-cell last column. So, to find the gf for floorsitters, it is enough to read off the coefficient of u^1 on the right-hand side of (4). That is easy: dividing the rhs of (4) by u, and then putting $u = 0$, we obtain

$$F(1) = \frac{\sum_{i=1}^{\infty}(-1)^{i-1}x^i y q^{\binom{i+1}{2}} \prod_{\ell=1}^{i-1}(1-y+yq^{\ell})/(q)_{i-1}}{\sum_{i=0}^{\infty}(-1)^i x^i q^{\binom{i+1}{2}} \prod_{\ell=1}^{i-1}(1-y+yq^{\ell})/(q)_i}, \tag{5}$$

where $F(1)$ is, of course, the gf for floorsitters.

Next on our agenda are 1-floorsitters. They include the single-celled polyomino and, in addition to it, just the polyominoes produced by the following rule: take an arbitrary floorsitter and continue it with a new single-celled column. The gf for 1-floorsitters is therefore

$$xyq + F(1)xq = \frac{\sum_{i=0}^{\infty}(-1)^i x^{i+1} y q^{\binom{i+2}{2}} \prod_{\ell=1}^{i-1}(1-y+yq^{\ell})/(q)_i}{\sum_{i=0}^{\infty}(-1)^i x^i q^{\binom{i+1}{2}} \prod_{\ell=1}^{i-1}(1-y+yq^{\ell})/(q)_i}. \tag{6}$$

In (6), the gf for 1-floorsitters has three variables: x is the horizontal semiperimeter, y the vertical semiperimeter and q the area. However, there is no problem in introducing two more variables, d the diagonals and s the floor-touching diagonals. Indeed, if P is a 1-floorsitter, then bottoms of P's columns are also bottoms of P's diagonals and *vice versa*. Hence the number of columns of $P =$ the number of diagonals of $P =$ the number of floor-touching diagonals of P.

So, to express our three-plus-two-variable gf, say $f_1(s)$, we just need to substitute dsx for x in (6). This gives us the case $j = 1$ of the following proposition.

Proposition 3. *For* $j \in N$, *the product* $f_1^{[j]}(s) := f_1(s) f_1(qs) \cdots f_1(q^{j-1}s)$ *can be expressed as*

$$f_1^{[j]}(s) = d^j s^j x^j y^j q^{\binom{j+1}{2}} \frac{g(q^j s)}{g(s)},$$

where

$$g(s) = \sum_{i=0}^{\infty} \frac{(-1)^i d^i s^i x^i q^{\binom{i+1}{2}} \prod_{l=1}^{i-1}(1-y+yq^l)}{(q)_i}. \tag{7}$$

Proof. The case $j=1$ is already established. For $j \geqslant 2$, the product $f_1^{[j]}(s)$ telescopes and is therefore easy to compute. Indeed,

$$f_1^{[j]}(s) = dsxyq\frac{g(qs)}{g(s)}dqsxyq\frac{g(q^2s)}{g(qs)}dq^2sxyq\frac{g(q^3s)}{g(q^2s)}\cdots dq^{j-1}sxyq\frac{g(q^js)}{g(q^{j-1}s)}$$

$$= d^js^jx^jy^jq^{\frac{j(j+1)}{2}}\frac{g(q^js)}{g(s)}. \qquad \square$$

After introducing some notations, we complete this section with an interesting (even if not indispensable) proposition.

Let $\mathscr{D}$ stand for the set of all ddc-polyominoes, and let $\mathscr{F}_j$ stand for the set of j-floorsitters. For $P \in \mathscr{D}$, we shall write:

di(P):=number of diagonals of P,

ft(P):=number of floor-touching diagonals of P,

h(P):=horizontal semiperimeter of P,

v(P):=vertical semiperimeter of P,

ce(P):=number of cells of P.

Now, $f_1^{[j]}(s)$ is not merely a "q-power".

Proposition 4. *For every $j \in N$, $f_1^{[j]}(s)$ is the gf for j-floorsitters.*

Proof. For $j=1$, there is nothing to prove.

Suppose the assertion holds for $j=m$.

Induction step. Let a *big diagonal* be a diagonal consisting of at least two cells.

Let $P \in \mathscr{F}_{m+1}$. Let D_- be the last one-celled diagonal of P, and let D_+ be the diagonal immediately following D_-. The diagonal D_+ is big, but is anyway contained in the shadow of D_-. Accordingly, D_+ has exactly two cells.

Let L be the figure formed by those diagonals of P which occur not later than D_-. Let T be the figure formed by those diagonals of P which occur not earlier than D_+.

The figure L is no doubt a 1-floorsitter.

Next, consider the horizontal line situated one unit above the floor of P. That line divides the figure T into two parts. The upper part (say U) is an element of $\mathscr{F}_m$, while the lower part (say R) is just a row of cells (see Fig. 2). We have di$(P)=$di$(L)+$di(U) together with similar decompositions for ft$(P), h(P)$ and $v(P)$. On the other hand, since ce$(R)=$ft(U), we have ce$(P)=$ce$(L)+$ce$(U)+$ft(U).

In addition, the mapping $P \mapsto (L,U)$ is a bijection between the set $\mathscr{F}_{m+1}$ and the Cartesian product $\mathscr{F}_1 \times \mathscr{F}_m$.

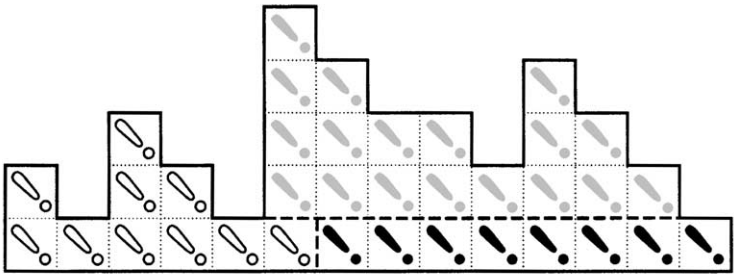

Fig. 2. A 4-floorsitter decomposed into a 1-floorsitter (white !'s), a 3-floorsitter (gray !'s), and a row of cells (black !'s).

Now it only remains to collect information together. As a result, we find the gf for $(m+1)$-floorsitters to be

$$\begin{aligned} & f_1(s) \cdot (\text{the gf for } m\text{-floorsitters, but with } qs \text{ in place of } s) \\ & \quad = f_1(s) f_1^{[m]}(qs) = f_1^{[m+1]}(s). \qquad \square \end{aligned}$$

5. All ddc-polyominoes

Our gf for all ddc-polyominoes is denoted as $D(s)$. In $D(s)$, the variables have the same names and roles as in $f_1(s)$.

The next proposition is something like the heart of this paper.

Proposition 5. *The gf $D(s)$ satisfies the equation*

$$D(s) = f_1(s) + \frac{x^{-1}}{1-qs} f_1(s) D(1) - \frac{x^{-1}(1-x+xqs)}{1-qs} f_1(s) D(qs). \tag{8}$$

Proof. Setting $d = x = y = 1$ makes the proof more clear and not much less general. So let only s and q survive! Instead of (8), we now have the equation

$$D(s) = f_1(s) + \frac{1}{1-qs} f_1(s) D(1) - \frac{qs}{1-qs} f_1(s) D(qs). \tag{8a}$$

Consider the right-hand side (rhs) of (8a). The first term being self-explanatory, we proceed to the rest.

Let P be an element of $\mathscr{D} \backslash \mathscr{F}_1$, the set of ddc-polyominoes which are not 1-floorsitters. Clearly, some diagonals of P (for example, the first one) are both one-celled and floor-touching. Among such diagonals, let D_- be the one which occurs last.

Let L be the "left factor" of P ending with the diagonal D_- . As in the proof of Proposition 4, L is a 1-floorsitter. Next, consider the horizontal line situated one unit above the floor of P. That line divides the figure $P \backslash L$ into two parts. After a little

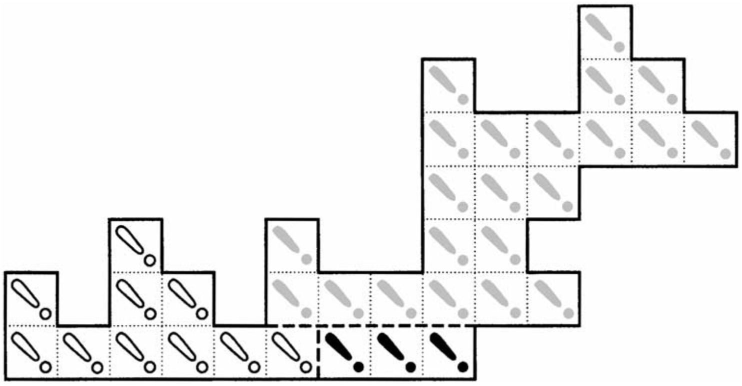

Fig. 3. Not being a 1-floorsitter, this ddc-polyomino decomposes into a 1-floorsitter (white !'s), a general ddc-polyomino (gray !'s), and a row of cells (black !'s).

thought, the upper part (say U) is seen to be a ddc-polyomino. On the other hand, the lower part (say R) is either the empty set or a finite row of cells (see Fig. 3). Needless to say, $\text{ce}(P)=\text{ce}(L)+\text{ce}(U)+\text{ce}(R)$ and $\text{ft}(P)=\text{ft}(L)+\text{ce}(R)$. Furthermore, it must be that $\text{ce}(R)\leqslant \text{ft}(U)$, because otherwise P would have a poor choice: to lie in $\mathscr{F}_1$ or not to lie in $\mathscr{D}$.

Can we recover the polyomino P once it is decomposed into L, U and R? Yes, here is the recipe:

(1) place U so that its source cell be the upper neighbor of the target cell of L,
(2) if R is not empty, place R so that its leftmost cell be the right neighbor of the target cell of L,
(3) then take the union $L\cup U\cup R$.

Thus, our decomposition is a bijection between $\mathscr{D}\backslash\mathscr{F}_1$ and the set $\{(L,U,R): L\in\mathscr{F}_1, U\in\mathscr{D},\ R$ is either the empty set or a row with $\leqslant \text{ft}(U)$ cells$\}$.

Putting the pieces together, we now obtain what follows. The gf for the set $\mathscr{D}\backslash\mathscr{F}_1$ is

$$\sum_{L\in\mathscr{F}_1}\sum_{U\in D}\sum_{i=0}^{\text{ft}(U)} q^{\text{ce}(L)+\text{ce}(U)+i}s^{\text{ft}(L)+i}$$

$$=\left[\sum_{L\in\mathscr{F}_1} q^{\text{ce}(L)}s^{\text{ft}(L)}\right]\sum_{U\in D} q^{\text{ce}(U)}\sum_{i=0}^{\text{ft}(U)} q^i s^i$$

$$=f_1(s)\sum_{U\in D} q^{\text{ce}(U)}\frac{1-(qs)^{\text{ft}(U)+1}}{1-qs}$$

$$= \frac{1}{1-qs} f_1(s) \sum_{U \in D} [q^{\mathrm{ce}(U)} - qsq^{\mathrm{ce}(U)}(qs)^{\mathrm{ft}(U)}]$$

$$= \frac{1}{1-qs} f_1(s)[D(1) - qsD(qs)]$$

$$= \frac{1}{1-qs} f_1(s)D(1) - \frac{qs}{1-qs} f_1(s)D(qs),$$

as required. □

Let $D(d,x,y,q)$ be another name for $D(1)$.

Theorem 1. *The gf for all ddc-polyominoes is given by*

$$D(d,x,y,q) = dxy$$

$$\frac{\sum_{i,j=0}^{\infty}(-dx)^i(-dy)^j q^{\binom{i+j+2}{2}}(q)_i^{-1}(q)_j^{-1}[\prod_{k=1}^{j}(1-x+xq^k)][\prod_{\ell=1}^{i-1}(1-y+yq^\ell)]}{\sum_{i,j=0}^{\infty}(-dx)^i(-dy)^j q^{\binom{i+j+1}{2}}(q)_i^{-1}(q)_j^{-1}[\prod_{k=1}^{j-1}(1-x+xq^k)][\prod_{\ell=1}^{i-1}(1-y+yq^\ell)]}. \tag{9}$$

Proof. We first iterate (8) in the usual way. This essentially means that we make a copy, say (C), of Eq. (8),

then we replace the term $D(qs)$, which Eq. (8) involves, with the case $s = qs$ of the rhs of (C),

then we replace the term $D(q^2s)$, which the equation obtained in the previous step involves, with the case $s = q^2s$ of the rhs of (C), and so on.

The iteration finished, we set $s = 1$ and obtain

$$D(1) = \frac{\sum_{j=0}^{\infty}(-1)^j x^{-j}\prod_{k=1}^{j}(1-x+xq^k)(q)_j^{-1} f_1^{[j+1]}(1)}{1 - \sum_{j=1}^{\infty}(-1)^{j-1}x^{-j}\prod_{k=1}^{j-1}(1-x+xq^k)(q)_j^{-1} f_1^{[j]}(1)}. \tag{10}$$

Proposition 3 now comes in handy. In fact, from (10) it allows us to obtain

$$D(1) = \frac{\sum_{j=0}^{\infty}(-1)^j x^{-j}\prod_{k=1}^{j}(1-x+xq^k)(q)_j^{-1} d^{j+1}x^{j+1}y^{j+1}q^{\binom{j+2}{2}} g(q^{j+1})/g(1)}{1 - \sum_{j=1}^{\infty}(-1)^{j-1}x^{-j}\prod_{k=1}^{j-1}(1-x+xq^k)(q)_j^{-1} d^j x^j y^j q^{\binom{j+1}{2}} g(q^j)/g(1)}$$

$$= dxy \frac{\sum_{j=0}^{\infty}(-1)^j x^{-j}\prod_{k=1}^{j}(1-x+xq^k)(q)_j^{-1} d^j x^j y^j q^{\binom{j+2}{2}} g(q^{j+1})}{\sum_{j=0}^{\infty}(-1)^j x^{-j}\prod_{k=1}^{j-1}(1-x+xq^k)(q)_j^{-1} d^j x^j y^j q^{\binom{j+1}{2}} g(q^j)}.$$

At this point, we insert (7), and the result quickly rearranges into formula (9) above. □

We knew it all along (because it is geometrically obvious) that the function D is symmetric in x and y. But the following fact is nevertheless worth pointing out.

Fact 1. *The relation* $D(d,x,y,q)=D(d,y,x,q)$ *may readily be seen from formula* (9).

Proof. The symmetry of the denominator is pretty obvious: when x and y exchange their places, the two swaps $i \leftrightarrow j$ and $k \leftrightarrow l$ restore the original expression.

The numerator of (9) seems to be tougher, but in reality it is (almost) not. Namely, the ith term of that numerator, say term N_i, may be written as $N_i(1-y+yq^i)$ plus $N_i y(1-q^i)$. In this way, the numerator splits into two double sums. On both of them, we can operate just like on the denominator. □

With x and y set equal to 1, formula (9) looks a good deal simpler.

Corollary 2. *We have*

$$D(d,1,1,q)=d\,\frac{\sum_{i,j=0}^{\infty}(-d)^{i+j}q^{i^2+(j+1)^2+i(j+1)}/(q)_i(q)_j}{\sum_{i,j=0}^{\infty}(-d)^{i+j}q^{i^2+j^2+ij}/(q)_i(q)_j}. \tag{11}$$

The less standard the derivation, the more important it is to verify the answer. Hence we verified (and found to be correct) formula (9) up to the terms in d^8, and formula (11) up to the terms in d^{10}. To do so, we resorted to *Maple* and *BASIC*, and we also recalled our [9] bijection between ddc-polyominoes and $\frac{1}{2}$-*good paths*.

Acknowledgements

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References

[1] V.K. Bhat, H.L. Bhan, Y. Singh, Enumeration of directed compact site animals in two dimensions, J. Phys. A: Math. Gen. 19 (1986) 3261–3265.
[2] M. Bousquet-Mélou, A method for the enumeration of various classes of column-convex polygons, Discrete Math. 154 (1996) 1–25.
[3] M. Bousquet-Mélou, Percolation models and animals, European J. Combin. 17 (1996) 343–369.
[4] M. Bousquet-Mélou, Mémoire d'Habilitation, Report No. 1154-96, LaBRI, Université Bordeaux I, 1996.
[5] M.-P. Delest, J.-M. Fédou, Exact formulas for fully diagonal compact animals, Report No. 89-06, LaBRI, Université Bordeaux I, 1989.
[6] I. Dutour, Grammaires d'objets: énumération, bijections et génération aléatoire, Ph. D. Thesis, LaBRI, Université Bordeaux I, 1996.
[7] S. Feretić, Novi rezultati u prebrojavanju poliomina, Ph. D. Thesis, University of Zagreb, 1998.
[8] S. Feretić, D. Svrtan, On the number of column-convex polyominoes with given perimeter and number of columns, in: A. Barlotti, M. Delest, R. Pinzani (Eds.), Fifth FPSAC proceedings, Firenze 1993, pp. 201–214.
[9] S. Feretić, D. Svrtan, Combinatorics of diagonally convex directed polyominoes, Discrete Math. 157 (1996) 147–168.

[10] I. Gessel, A noncommutative generalization and q-analog of the Lagrange inversion formula, Trans. Amer. Math. Soc. 257 (1980) 455–482.

[11] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, Addison–Wesley, Reading, MA, 1989.

[12] N. Inui, M. Katori, G. Komatsu, K. Kameoka, The number of directed compact site animals and extrapolation formula of directed percolation probability, J. Phys. Soc. Japan 66 (1997) 1306–1309.

[13] J.-G. Penaud, Animaux dirigés diagonalement convexes et arbres ternaires, Report No. 90–62, LaBRI, Université Bordeaux I, 1990.

[14] J.-G. Penaud, Arbres et animaux, Mémoire d'Habilitation, LaBRI, Université Bordeaux I, 1990.

[15] V. Privman, N.M. Švrakić, Exact generating function for fully directed compact lattice animals, Phys. Rev. Lett. 60 (1988) 1107–1109.

[16] H.N.V. Temperley, Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules, Phys. Rev. 103 (1956) 1–16.